completion and diminution, The book on the rule of double false position, The book of surveying and geometry, The book of the adequate (possibly in arithmetic).[1] Such is the account of the life and works of Abu Kamil as given in the Kitab al-Fihrist (987 A. D.) of An-Nadim, who includes this writer in his list of the later (*i. e.*, nearly contemporary with An-Nadim) reckoners and arithmeticians. Aside from the works included in this list a treatise on the mensuration of the pentagon and the decagon exists in Latin and Hebrew translations, and a further arithmetical work is preserved in the original Arabic as well as in Latin and Hebrew. The work on the pentagon and decagon has been published in Italian by Sacerdote,[2] and in German by Suter.[3] The arithmetical work, in which he deals with first degree indeterminate equations, has been translated by Suter.[4] Other information about the life and activity of this great Arab we do not have. The titles mentioned indicate that, like so many of the later scientists in Europe, Abu Kamil was interested in magic, doubtless including astrology, and in omens.

The Sources of Information. The manuscript upon which I have based my study of Abu Kamil's algebra is the Paris manuscript 7377A of the Bibliothèque nationale.[5] The treatise covers folios 71^{v}–93^{v}, and contains between 35,000 and 50,000 words.[6]

In the opening sentence the author refers to his famous predecessor in this field, Mohammed ibn Musa al-Khowarizmi, and a little later again refers to the algebra of the same writer. Several references are made to Euclid but aside from these two names no others have been noted in this algebra.

Quotations from the Manuscript. The following translation of selected passages from the algebra is a very free one, as an attempt is made to preserve the mathematical significance. Additions to clarify the meaning, and modern notations, are put in parentheses.

The first thing which is necessary for students of this science is to understand the three species which are noted by Mohammed ibn Musa al-Khowarizmi in his book. These are *roots*, *squares* and *number*. A root is anything which can be multiplied by itself, composed of one, and numbers above one, and fractions. A square is that which results from the multiplication by itself of root, composed of units, and units and fractions. A number is a quantity by itself which does not have the name of root or square, but is proportional to the number of units which it contains. These three species are proportional to each other by turn in twos. Thus, squares equal to roots, squares equal to number, and roots equal to number ($ax^2 = bx$, $ax^2 = n$, $bx = n$).

An illustration of squares equal to roots is this:

A square is equal to five of its roots ($x^2 = 5x$). The explanation of this is that a square is five times the root of it. The root of the square is always the same as the number of the roots to which the square is equal. In this question this root is five and the square is twenty-five

[1] I translate from the German translation of the passages in the Fihrist dealing with mathematical scientists, by Suter, in *Abhandl. z. Geschichte d. math. Wissen.*, VI, 1–87.

[2] *Festschrift Steinschneiders*, Leipzig, 1896, 169–194.

[3] *Bibliotheca mathematica*, X, third series, 15–42.

[4] *Das Buch der Seltenheiten der Rechenkunst*, in *Bibl. math.*, 3d ser., XI, 100–120.

[5] I am indebted to the librarian for permission to have photographic reproductions made.

[6] To give an idea of the extent of this work, if the Latin text were printed in the MONTHLY it would take between 75 and 100 pages.

which is the same as five of its roots. That the root of the square is the same as the number of the root this I will explain.

We place for the square a square surface *abgd* whose sides are *ab*, *bg*, *gd*, and *da* (see Fig. 1). Of this square any side multiplied by one in number is the same as the area of a surface whose length is the side or root of the square. The side *ab* multiplied by one which is *be* gives the surface *ae* which is the root of the square *ag*. The surface *ag* is the same as five roots or five times the root of itself. Therefore, it is five times the surface *ae*. Divide then, the surface *ag* by equidistant lines into five equal parts, that is the surfaces *ae*, *ck*, *sr*, *nh*, and *mg*. The lines *be*, *ek*, *kr*, *rh* and *hg* are equal. Moreover the line *be* is one. Therefore the line *bg* will be five which is the root of the square and the square itself is twenty-five which is the surface *ag*. This is what we wished to explain.

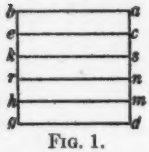

FIG. 1.

If it is proposed that one half a square should equal 10 roots ($\frac{1}{2}x^2 = 10x$) then the whole square is 20 roots. The root of the square is 20 and the square 400. Similarly if it is proposed that 5 squares equal 20 roots ($5x^2 = 20x$) then the root of the square is 4 and the square is 16. And whether more than a square or less than a square (is proposed) you reduce to 1 square. You operate in the same manner with that which is connected with the square, the roots.

A square which is equal to a number is illustrated:

A square equals 16 ($x^2 = 16$). Therefore, it itself is 16 and the root of it is 4. Similarly if given 5 squares equal 45 drachmas ($5x^2 = 45$), then 1 square is a fifth of 45 which is 9 and the root of it is 3. . . . So whether more or less than a square (is given) it is reduced to 1 square and similarly is treated that which is joined or equal to them of numbers.

Roots which are equal to numbers are illustrated:

A root equals 4 ($x = 4$). Therefore, the root itself is 4 and the square is 16. Similarly again if it is proposed that 5 roots should equal 30 ($5x = 30$) then the root is equal to 6 and the square 36. . . .

We construct now these three species, that is roots, squares and numbers, joined by twos and proportional to the third. This gives, a square and roots equal to number, a square and number equal to roots, and roots and number equal to squares.

(In modern notation $ax^2 + bx = n$, $ax^2 + n = bx$, $bx + n = ax^2$.)

First Type of Quadratic Equation. An illustration of squares and roots equal to number is the following:

A square and 10 roots equal to 39 drachmas ($x^2 + 10x = 39$). The explanation of this is that there is some square to which if you add the same as 10 of its roots the number of it results in 39 drachmas. In this question there are two ways (methods) of which one leads you to know the root of the square and the other to know the square of the root. We will set forth and explain each of these by a geometrical figure which will be understood by those who understand the book of Euclid, that is the Elements. The method which leads us to know the root of the square has been narrated by Mohammed ibn Musa al-Khowarizmi in his book. This is that you divide the roots in half, that is in two equal parts. In this question this gives 5. You multiply this by itself, giving 25, which you add to 39. This gives 64 of which number you take the root which is 8; from this you take the half of the roots, that is 5, and there remains 3, which is the root of the square. The square is 9.

The method indeed which leads you to know the square is that you multiply 10 roots by itself, giving 100. This you multiply by 39, which is equal to the square and roots, giving 3,900. Now divide 100 in halves, giving 50, which you multiply by itself, giving 2,500. This add to 3,900, giving 6,400, and of this number take the root, which is 80. This you subtract from 50, which is the half of 100, and from 39, which is equal to the square and the roots, together giving 89, and 9 remains, which is the square.[1]

[1] The text to this point is found in the Paris MS. on folio 71^v–72^a; given in my article in the *Bibliotheca Mathematica, loc. cit.*, p. 42–44.

In modern notation the steps of this solution are indicated as follows, in which both the general case and this special problem are given in parallel columns.

(1)	$x^2 + bx = n,$	$x^2 + 10x = 39,$
(2)	$b^2x^2 + b^2x = nb^2,$	$100x^2 + 1000x = 3{,}900,$
(3)	$b^2x^2 + b^2x + \left(\frac{b^2}{2}\right)^2 = nb^2 + \frac{b^4}{4},$	$100x^2 + 1000x + 2500 = 6{,}400,$
(4)	$bx + \frac{b^2}{2} = +\sqrt{nb^2 + \frac{b^4}{4}},$	$10x + 50 = 80.$

Only the positive root is taken, as the negative root leads to the negative solution of the equation. Subtracting (4) from (1), member for member,

$$x^2 - \frac{b^2}{2} = n - \sqrt{nb^2 + \frac{b^4}{4}}, \qquad x^2 - 50 = 39 - 80,$$

$$x^2 = n + \frac{b^2}{2} - \sqrt{nb^2 + \frac{b^4}{4}}, \qquad x^2 = 50 + 39 - 80,$$

$$x^2 = 9.$$

Following this the problem is proposed:

$$2x^2 + 10x = 48.$$

This is a problem which is also given by Al-Khowarizmi and the same is true of many further problems given by Abu Kamil. The latter author gives the method of solution presented by Al-Khowarizmi and also the method, above illustrated, leading to the value of x^2 directly. Further than this he presents the geometrical solution for the first method as given by his more famous predecessor. Abu Kamil adds the geometrical solution corresponding to his own method, the second of those given above. This seems to be worth presenting here.

In order to show you the method (the door, literally) which leads to a knowledge of the square we place for the square the line *ab* (see Fig. 2). To this we add 10 roots of itself which are represented by the line *bg*, giving the line *ag*, which is 39. We wish to know then the value of the line *ab*. We construct then upon the line *bg* a square surface, which is the surface *degb*. This then will be $100x^2$ (in modern notation), that is, 100 times the line *ab*. . . . We construct then the surface *ah* the same as the square surface *be*, that is equal to the square, which is the same as the multiplication of the line *ab* in one of the units which it contains taken 100 times. Then we complete the surface *an* which is 3,900, since the line *ag* is 39 and *am* is 100, which lines contain this surface. Moreover the surface *ah* is the same as the surface *be*. Therefore, the surface *dn* is 3,900, which is produced by the multiplication of the line *ne* by *eg*, as the line *eg* is the same as the line *ed* and the line *gn* is 100 since it is the same as the line *am*. Therefore we divide the line *gn* in two equal parts at the point *l*, to which line so divided the line *ge* is added in length. Therefore, the multiplication of the whole line *ne* by *eg* together with the product of the line *gl* by itself is the same as the product of the line *le* by itself, as Euclid says in the second book[1] of his Elements. The product of *ne* by *eg* is 3,900 and the product of *gl*

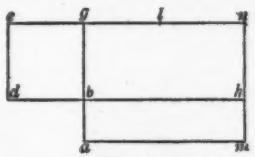

FIG. 2.

[1] Euclid's *Elements*, II, 6, "If a straight line is bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line, together with the square on the half, is equal to the square on the straight line made up of the half and the added straight line."

by itself is 2,500, which being combined make 6,400, which is the same as the product of the line *le* by itself. Therefore, the product of the line *le* by itself is 6,400, of which the root is 80, which is the line *le*. But the line *ge* is the same as the line *gb*, and so the line *lg* and the line *gb* make 80. When we take away the line *lg* and the line *dg*, which are 80, from the lines *ag*, 39, and *gl*, 50, which make 89, the line *ab* remains, which is 9. This is the square and that is what we desired to explain.[1]

Second Type of Quadratic Equations. The first problem which Abu Kamil presents of the second type of composite quadratic equations is again the same as that given by Al-Khowarizmi,

$$x^2 + 21 = 10x.$$

For this type of equation also Abu Kamil presents two solutions, the one leading directly to the value of the root of the equation and the other to the value of the square of the root. Further the author notes that there are two solutions for both the root and the square in this type of quadratic equation. The reason that this was noted by both of these Arabic writers is that in this type of equation the two roots are positive, whereas in the other two types the one root is positive and the other root is negative. Negative quantities as such are not accepted by these Arabic mathematicians and this, of course, accounts for the three types of quadratic equations. Express mention is also made of the fact that if the square of half the coefficient of the roots is less in magnitude than the constant then the problem is impossible, from the Arabic standpoint, or contradictory. Further also, the author mentions that when this square is equal to the given constant the root of the equation is the same as one half the given coefficient.

In the above problem,

$$x^2 + 21 = 10x,$$

the analytical solution leading to the value of the square of the unknown quantity proceeds in substance as follows:

Multiply 10 by 10, giving 100. Multiply this by 21, giving 2,100. Then take half of 100, giving 50, which you multiply by itself, giving 2,500. From this subtract 2,100, leaving 400 of which the root is 20, which you subtract from 50, the half of 100, leaving 30. From this you subtract 21, leaving 9, which is the square. And if you wish, add 20 to 50, giving 70, from which you subtract 21, leaving 49, which is the square and the root of it is 7.[2]

Al-Khowarizmi gave the geometrical figure corresponding only to the smaller root of the equation. Thus in the equation, $x^2 + n = bx$, in which

$$x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - n},$$

one of these roots, when both are real, is necessarily less than $b/2$ and the other is greater. Abu Kamil gives the geometrical explanation corresponding to both, thus completing the work of Al-Khowarizmi. Thus he says that when you assume that the square is less (in numerical value) than the number which is given, then the solution appears by subtraction, and when you assume the same

[1] In the manuscript folio 72^v–73^r; in *Bib. Math., loc. cit.*, p. 46.

[2] Manuscript folio, 73^r; *Bib. Math., loc. cit.*, p. 47.

greater than the constant the solution appears by addition. Further he gives a geometrical solution for the type in which the square of the coefficient of x is equal to the given constant. The figure in this case consists simply of two equal squares with a side in common. We follow with a demonstration, original with Abu Kamil, leading to value of x^2 in the equation, $x^2 + 21 = 10x$.

. . . Place the line *ab* to represent x^2 (Fig. 3), and we add to it the drachmas which accompany it, 21, and let it be the line *bg*. Therefore, the line *ag* is 10 roots of the line *ab*. We construct then upon the line *ag*, a square surface, the square *aged* and this is 100 times the line *ab* . . . since the line *ag* is 10 roots of the line *ab*, and 10 roots multiplied by itself gives 100 squares (x^2). We construct then the surface *ah* equal to the square *aged* and let one side of it be the line *ab*. Therefore, the other side *bh* is 100. Then we complete the surface *an* and therefore the surface *bn* is 2,100 since the line *bg* is 21 and the line *gn* 100. We construct then the surface *my* equal to the surface *ae*, but the surface *ae* is equal to the surface *ah*. Take away, then, the surface *th* and there remains the area *cn* equal to the area *ca*. Adding then the surface *by*, the whole surface *ay* is equal to the surface *bn*, which is 2,100. Therefore the area *ay* is 2,100 which is the product of *gy* by *yn*, since *yn* is equal to *yt*, since the area *tn* is a square. We divide then the line *gn* in two equal parts at the point *l*. Therefore the line *ng* is divided into two equal parts by the point *l* and into two unequal parts by the point *y*. Therefore the product of *gy* by *yn* together with the square upon *ly* is the same as the product of *ln* by itself.[1] But *ln* by itself makes 2,500, since it is 50, and the product of *gy* by *yn* is 2,100. The square on the line *ly* is then left as 400 and the line *ly* 20. But *ln* is 50 and as it (*ly*) is to be subtracted the line *yn* is left as 30. Since the line *bg* is 21 the remaining line *ab* is 9 which is the square. This is what we wished to explain.[2]

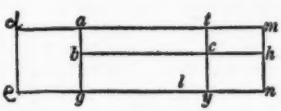

FIG. 3.

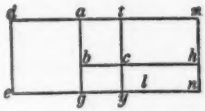

FIG. 4.

Having given the explanation "by subtraction," that is to say, when the square root of 2,500 − 2,100 is subtracted from 50, Abu Kamil proceeds to explain "by addition." *gn* is again divided into two equal parts but the point of division falls between *y* and *n* (Fig. 4). This corresponds to the assumption that the square is greater than the number which accompanies it. This has also been noted above.

The product of *gy* by *yn* together with the square of *yl* equals the product of *lg* by itself. *lg* multiplied by itself gives 2,500 and the line *yn* by *yg* gives 2,100. Therefore the square of *ly* is 400 and *ly* is 20. Since *ln* is 50, the whole line *yn* is 70. But *yn* is the same as *yt* and *yt* equals *ag*. Therefore *ag* is 70. But *bg* is 21 whence *ab* is 49 which is the square. This is what we desired to explain.[3]

In the third type of quadratic equations Abu Kamil again follows Al-Khowarizmi in taking the equation $3x + 4 = x^2$. Three explanations are presented for finding the root of the square. The first is equivalent to the demonstration presented by Al-Khowarizmi but the figure is not completed. Instead of this the proof is made to depend upon the second book of Euclid. The second gives

[1] This is by Euclid VI, 5, "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, together with square on the straight line between the points of section, is equal to the square on the half."

[2] Manuscript folio, 74r; *Bib. Math.*, *loc. cit.*, pp. 48–49.

[3] Manuscript folio, 74r; *Bib. Math.*, *loc. cit.*, p. 49.

the completed figure and explains as in the algebra of Al-Khowarizmi. In his third figure (Fig. 5) the square *abdg* is placed to represent x^2 which by the conditions of the problem equals $3x + 4$. Then the lines *ah* and *yd* are taken $1\frac{1}{2}$ units in length and the rectangles completed cutting off from the original square $3x$ less the little square, $1\frac{1}{2}$ on a side, which these two rectangles have in common. Hence the square which remains *hnyz* is $4 + (1\frac{1}{2})^2$ or $6\frac{1}{4}$. It follows that *gh* is $2\frac{1}{2}$ and *ay* (or x) is 4.

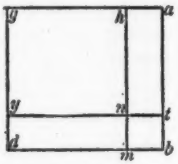

FIG. 5.

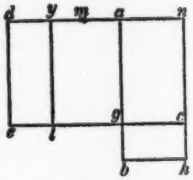

FIG. 6.

To arrive at the value of x^2 geometrically Abu Kamil places (Fig. 6) *ab* for the square and cuts off *gb* equal to 4, leaving *ag* equal to the three roots. Then he continues:

Upon *ag* a square is constructed *aged* whose area is $9x^2$ or 9 times the line *ab*. *ah* is constructed equal to the square *ae*. From this it follows that *an* is 9 and that *gc* which equals *an* is 9. Since *bg* is 4 the area of *gh* is 36. Take *ay* equal to *an* and draw *yl* parallel to *de*. The area *yg* equals *ac* and the area *ae* equals *ah*. Hence *ye* equals *cb* which is 36. But *ya* is 9 since *an* is 9. Divide *ay* into two equal parts by the point *m*. Now *yd* is added in length to *ay*. Hence *ad* by *dy* plus *ym* multiplied by itself equals *md* multiplied by itself, as EUCLID says in his second book. But *ad* by *dy* is *ye* or 36. *ym* multiplied by itself is $20\frac{1}{4}$. Adding you obtain $56\frac{1}{4}$. Therefore *md* multiplied by itself gives $56\frac{1}{4}$ and *md* is $7\frac{1}{2}$. But *am* is $4\frac{1}{2}$ and *gb* is 4. Hence *ab* is 16 and this is what we desired to demonstrate.[1]

Abu Kamil summarizes the results of the algebra up to this point, stating that all questions that can be solved by *algebra and almuchabala*, that is, restoration and opposition, must reduce to one or the other of the six types of equations mentioned.

Multiplication and Division. In the next section the multiplication and division of algebraic quantities is considered. Binomials, involving both addition and subtraction, are given particular consideration. The writer uses *res* and *radix*, indifferently, for the first power of the unknown. This is combined by addition or subtraction with a pure number. The second power of the unknown is designated by *census*, but this word is sometimes used simply for any unknown quantity as in problems below. Similar usage obtains in the work of Leonard of Pisa and in the *Liber augmenti et diminutionis* by one Abraham, published by Libri, *Histoire*, I, 304–371.

After a general statement about multiplication which summarizes the work presented, Abu Kamil continues with a geometrical explanation of the problems. Thus a figure containing four small squares is used to illustrate the fact that $2x$ multiplied by $2x$ gives $4x^2$. The next problem is $3x$ multiplied by 6 drachmas.

[1] Manuscript folio, 75^r; *Bib. Math.*, *loc. cit.*, pp. 49–50.

In this section the constant is referred to drachmas although in other places the word *numerus* is also used. Figures are shown for $10 + x$ by x, $10 - x$ by x, $10 + x$ by $10 - x$ and other similar examples.

Radical Expressions. The addition and subtraction of radicals, involving quadratic irrationalities only, is effected by means of the relations given in modern symbols by the equalities $\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm 2\sqrt{ab}}$. Thus, to subtract the square root of 8 from the square root of 18 the rule is simply: "subtract 24 from 26, as you know, and 2 remains. The root of this is the root of 8 subtracted from the root of 18." This problem, by the way, is given by Al-Karkhi. Leonard of Pisa[1] employs the same method but uses 18 and 32, instead of 8 and 18, combining them by addition and subtraction in the same way as Abu Kamil.

Al-Karkhi[2] proceeds to the consideration of cube roots in a similar way but our author takes up for a final problem in roots the addition of $\sqrt{10}$ to the $\sqrt{2}$ which, he states, does not lead to a simple result since $10 : 2 = 5$ which is not a square anld aso $2 : 10 = \frac{1}{5}$ which is not a square. This section concludes with the statements that $\sqrt{10} + \sqrt{2} = \sqrt{12 + 2\sqrt{20}}$ and $\sqrt{10} - \sqrt{2} = \sqrt{12 - 2\sqrt{20}}$.

Problems. The section dealing with problems is introduced in much the same way that Al-Khowarizmi does the corresponding section, and many of the problems are taken from the older work. The first problem is to divide ten into two parts such that the square of the larger part should equal $1\frac{1}{2}$ times the product of the two parts. This leads to the equation

$$x^2 = 1\tfrac{1}{2}x(10 - x),$$

or

$$2\tfrac{1}{2}x^2 = 15x.$$

Thus the problem leads to the first of the six types of quadratic equations.

The above problem is the second presented by Leonard of Pisa[3] in the section with the heading: *Expliciunt introductiones algebre et almuchabale. Incipiunt questiones eiusdem.* Furthermore this problem does not appear in Al-Karkhi[4] in the list given by Woepcke. The conclusion is reasonable that Leonard had some source of information concerning the algebra of Abu Kamil other than Al-Karkhi's work, and even more particularly in view of the long series of problems found in Leonard's work which correspond to those given by Abu Kamil. The second and third questions here are found to be the same as the following two questions in Leonard of Pisa,[5] but with slightly different numbers. Thus instead of $6\frac{1}{4}x^2 = 100$, Leonard and Al-Khowarizmi give $2\frac{7}{9}x^2 = 100$ and in place of $x/(10 - x) = 4$ as given by Abu Kamil and Al-Khowarizmi, Leonard sets this same fraction equal to $2\frac{1}{3}$. Our space does not permit an examination of all similarities of this kind.

[1] *Scritti di Leonardo Pisano*, published by Boncompagni, Vol. I: *Il liber abbaci*, Roma, 1857, p. 363–365.

[2] F. Woepcke, *Extrait du Fakhri*, Paris, 1853, p. 57–59.

[3] *Liber abbaci*, p. 410.

[4] Woepcke, loc. cit., p. 75–137.

[5] *Liber abbaci*, p. 410.

A problem given on folio 81^r is of particular interest because of its similarity to a problem of Al-Khowarizmi's[1] which has given rise to discussion. The problem is to divide 50 among some number of men and then to add three men and again distribute 50 (drachmas) among them (equally). In the second instance each man receives $3\frac{1}{2}\ \frac{1}{4}$ (meaning $3\ 1/2 + \frac{1}{4}$) drachmas less than in the first. Leonard states the corresponding problem as follows:

> Distribute 60 among (a certain number of) men and each will receive something. Add two men and among the... all again distribute 60 and then each man will receive $2\frac{1}{2}$ denars less than at first.

The statement of Abu Kamil is:

> And if we tell you, distribute 50 among (a certain number of) men and each receives a certain amount (*res*). Add three men and distribute again 50 drachmas among them. Each then receives $3\frac{1}{2}\ \frac{1}{4}$ drachmas less than before.

Both follow with a geometrical explanation.

Problems Involving Fractions. The notation of fractions employed by this translator of Abu Kamil resembles the peculiar system employed later by Leonard. Thus on fol. 83^r the square of $\frac{1}{2}$ and of $\frac{1}{9}$ are given in combinations as in the following translation:

. . . $\frac{1}{2}$ and $\frac{1}{9}$ which multiplied by itself, gives $\frac{1}{4}$ and $\frac{1\,1}{9\,9}$. To this we add 11 drachmas and $\frac{1}{9}$, giving 11 drachmas and $\frac{1}{4}$ and $\frac{1}{6}$ and $\frac{e\,1}{9\,2}$ and $\frac{e\,1}{9\,9}$. Of this we take the root or 3 and $\frac{1}{3}$ and $\frac{0\,1}{9\,2}$ . . . .

The fraction $\frac{1\,1}{9\,9}$ stands for $\frac{1}{9}$ plus $\frac{1}{81}$ whereas $\frac{e\,1}{9\,2}$ as well as $\frac{0\,1}{9\,2}$ represents simply $\frac{1}{18}$. So also Leonard[2] uses $\frac{5\,e}{8\,8}$ for $\frac{5}{64}$. This system is explained by Leonard[3] who writes for $\frac{1}{14}$ the form $\frac{1\,0}{2\,7}$ and $\frac{1\,5\,7}{2\,6\,10}$ for $\frac{7}{10}+\frac{5}{60}+\frac{1}{120}$. The difference is only in the order of reading.

Al-Khowarizmi gives several problems of this nature:[4]

> To find a square of which if one-third be added to three dirhems, and the sum be subtracted from the square, the remainder, multiplied by itself restores the square.

The solution involves the use of x (root) to represent the square. So also Leonard[5] in a similar problem states: ". . . place for that square x (*res*)." Our version of Abu Kamil has several problems of the same kind:

> And if we tell you there is a quantity (*census*) from which when you subtract $\frac{1}{3}$ of itself and 2 drachmas, the product of the remainder by itself gives the quantity and 24 drachmas.

The solution is obtained by placing x (*res*) for the quantity (*census*).

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[1] Libri, *Histoire des sciences mathématiques en Italie*, Vol. 1, Paris, 1838: "Liber Maumeti . . . de algebra et almuchabala," p. 286. Rosen, *The Algebra of Mohammed ben Musa*, London, 1831, p. 63–64. Al-Khowarizmi does not present any geometrical explanation.

[2] *Liber abbaci*, p. 447.

[3] *Liber abbaci*, p. 24, 25.

[4] Rosen, *loc. cit.*, p. 56, 57.

[5] *Liber abbaci*, p. 422.

The longest discussion of any algebra problem in Leonard is that of the following:[1]

Divide 10 in two parts, and divide the larger by the smaller, and the smaller by the larger. Sum the results of the division and this equals the square root of 5.

Leonard presents several solutions. In the first he arrives at the equation $\sqrt{5x^4 + 2x^2 + 100} = 20x + \sqrt{500x^2}$. The coefficient of x^2 is made equal to unity by multiplying through by $\sqrt{5} - 2$. The same equation is found in Abu Kamil and the treatment is the same. The value of x is $5 - \sqrt{225 - \sqrt{50000}}$ which is found by both writers. Leonard goes on to find the value of $\sqrt{225 - \sqrt{50000}}$ as $\sqrt{125} - 10$. Al-Khowarizmi and Al-Karkhi do not give this problem although they do give one upon which this is based, namely: to divide 10 into two parts such that the sum of each divided by the other is $2\frac{1}{6}$. Al-Karkhi[2] discusses four solutions.

Leonard of Pisa occasionally solves incorrectly, if this expression be permissible. Thus the problem:[3] To find a number such that, if the square root of 3 be added to it and then the square root of 2 be added, the product of the two sums will be 20. Algebraically $(x + \sqrt{3})(x + \sqrt{2}) = 20$. The product of the two binomials is given as $x^2 + 6 + \sqrt{12x^2} + \sqrt{8x^2}$. This leads to the incorrect value $\sqrt{19 + \sqrt{24}} - \sqrt{3} - \sqrt{2}$ for x. Our manuscript of Abu Kamil gives the correct value, $\sqrt{21\frac{1}{4} + \sqrt{1\frac{1}{2}} - \sqrt{6} - \sqrt{\frac{3}{4}} - \sqrt{\frac{1}{2}}}$, for x. The problems[4] following this in the algebra of Leonard, with the exception of the two final ones, are all taken from Abu Kamil and in the same order in which they occur in the work of the Arabic writer.

The Italian writer frequently uses an expression which might be supposed to refer to Abu Kamil: "Operate according to algebra,"[5] . . . "Operate therefore in this according to algebra, etc." However our manuscript employs a similar expression. Thus in a problem arriving at $16x^4 = 256x^2$, this Latin version of Abu Kamil reads: Fac secundum algebra in eis et erunt 16 census census equales 256 censibus. census census (x^1) ergo equatur 16 censibus et census equatur 16 dragmis, ergo res equatur 4 dragmis . . ." (fol. 91^r). In the discussion of the same problem Leonard concludes one form of solution with the words: "Age ergo in eis secundum algebra, et inuenies, census census equari 16 censibus: quare census est 16, et radix eius est 4, ut dictum est." Both writers present several solutions, including a geometrical one, of the problem in question which is to divide 10 into two parts such that if from the larger you subtract two of its roots and to the smaller add two of its roots the quantities are then equal. The expression *secundum algebra* refers to the Book of Algebra and Almucabala by Mohammed ben Musa.

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[1] *Liber abbaci*, p. 434–438.
[2] Woepcke, *loc. cit.*, p. 91, 92.
[3] *Liber abbaci*, p. 445.
[4] *Liber abbaci*, 445–459.
[5] *Liber abbaci*, 438.

Treatise on the Pentagon and Decagon. The algebra terminates with the first eight lines on fol. 93^v, and is followed by the work on the pentagon and decagon by the same author. Although this treatise on the pentagon and decagon by Abu Kamil is geometrical in its nature yet the treatment and the solutions are algebraical, including a fourth degree equation ($x^4 = 8000x^2 - \sqrt{51\,200\,000}$) as well as mixed quadratics with irrational coefficients. The twelfth problem is in the Latin: "Et si dicemus tibi trianguli equilateri et equianguli mensura est cum perpendiculari ipsius est 10 ex numero, quanta sit perpendicularis?"[1] This suggests the similar problems in Greek of unknown date and author presented by Heiberg and Zeuthen,[2] as well as the similar problems given by Diophantos. In these Greek problems also lines and areas are summed quite contrary to ancient Greek usage. A further point of interest is that in the equation $x^2 + 75 = 75x$, to which the solution of the thirteenth problem leads, Abu Kamil gives only one solution whereas in the algebra he recognizes that this equation has two positive roots. The opening sentence of this treatise on the pentagon and decagon makes reference to the algebra as immediately preceding it, which indeed is the fact in the Hebrew and Latin manuscripts that are preserved.

Conclusion. The algebra terminates with a general statement to the effect that by the methods taught in this book many more problems can be easily solved. In true Arabic fashion, the closing words are: " Whence praise and glory be to the only Creator."

Let us summarize the results of our study. The most important conclusion of this investigation of Abu Kamil's algebra is that Al-Karkhi and Leonard of Pisa drew extensively from this Arabic writer. Through them this man, though himself comparatively unknown to modern writers, exerted a powerful influence on the early development of algebra. Abu Kamil deserves somewhat the same recognition from modern mathematicians and historians of science as that which Leonard of Pisa and Al-Khowarizmi have received. We may hope that the future will be more just than the past in according to Abu Kamil a prominent place among the mathematicians of the middle ages.

A CURIOUS CONVERGENT SERIES.

By A. J. KEMPNER, University of Illinois.

It is well known that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges. The object of this Note is to prove that if the denominators do not

[1] "If we say to you that an equilateral and equiangular triangle, together with its altitude, is measured by 10, what is the altitude?"

[2] *Einige griechische Aufgaben der unbestimmten Analytik*, in *Biblioth. mathem.*, VIII, third series, 118–134.

include all natural numbers 1, 2, 3, $\cdots$, but only those numbers which do not contain any figure 9, the series converges. The method of proof holds unchanged if, instead of 9, any other figure 1, 2, $\cdots$, 8 is excluded, but not for the figure 0.

Proof: The series with which we deal is

$$\text{I.}\quad \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8}+\frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots+\frac{1}{28}+\frac{1}{30}+\cdots$$
$$+\frac{1}{38}+\frac{1}{40}+\cdots+\frac{1}{48}+\frac{1}{50}+\cdots+\frac{1}{58}+\frac{1}{60}+\cdots+\frac{1}{68}+\frac{1}{70}+\cdots$$
$$+\frac{1}{78}+\frac{1}{80}+\cdots+\frac{1}{88}+\frac{1}{100}+\cdots.$$

We form a new series

$$\text{II.}\qquad s = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

by the following rule: a_n is the sum of all terms in I of denominator d, where $10^{n-1} \leqq d < 10^n$. We have then

$$a_1 = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8},$$

$$a_2 = \frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots, \text{ the last term being } \frac{1}{88}, \text{ etc.}$$

Each term of I which forms part of a_1, is $\leqq 1$,

each term of I which forms part of a_2, is $\leqq 1/10$, and

each term of I which forms part of a_n, is $\leqq 1/10^{n-1}$.

We now count the number of terms of I which are contained in a_1, in a_2, $\cdots$, in a_n. Evidently a_1 consists of 8 terms, and $a_1 < 8 \cdot 1 < 9$. In a_2 there are, as is easily seen, less than 9^2 terms of I, and $a_2 < (9^2/10)$. Altogether there are in I less than $9^2 + 9$ terms with denominators under 100. Assume now that we know the number of terms in I which are contained in a_n to be less than 9^n, for $n = 1, 2, 3, \cdots, n$. Then, because each term of I which is contained in a_n is not greater than $1/10^{n-1}$, we have $a_n < (9^n/10^{n-1})$, and the total number of terms in I with denominators under 10^n is less than $9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9$. For $n = 1$ and $n = 2$ we have verified all this, and we will now show that if it is true for n, then $a_{n+1} < (9^{n+1}/10^n)$. a_{n+1} contains all terms in I of denominator d, $10^n \leqq d < 10^{n+1}$. This interval for d can be broken up into the nine intervals $\alpha \cdot 10^n \leqq d < (\alpha + 1)10^n$, $\alpha = 1, 2, \cdots, 9$. The last interval does not contribute any term to I, the eight remaining intervals contribute each the same number of terms to I, and this is the same as the number of terms contributed by the whole interval $0 < d < 10^n$, that is, by assumption, less than

$9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9$. Altogether, therefore, a_{n+1} contains less than $8(9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9) < 9^{n+1}$ terms of I, and, as each of these terms is not greater than $1/10^n$, we have $a_{n+1} < (9^{n+1}/10^n)$.

Our series is therefore

$$s = a_1 + a_2 + a_3 + \cdots < 9 + \frac{9^2}{10} + \frac{9^3}{10^2} + \cdots + \frac{9^{n+1}}{10^n} + \cdots = 90,$$

and I is certainly convergent.

The preceding proof also contains all material necessary to show the following:

Let M be any positive integer, and N_1 the number of positive integers $< M$ containing no figure 9, N_2 the number of positive integers $< M$ containing at least one figure 9, then

$$\lim_{M=\infty} N_1/N_2 = 0.$$

See problem 207 proposed on page 55 of this issue.

THE PERFECT MAGIC SQUARES FOR 1914.

By V. M. SPUNAR, Chicago, Ill.

Using Zerr's formula*

$$\frac{1914 - (n^3 + n)/2}{n} = \text{Integer},$$

the only values of n for 1914 are 3, 11, 29. (For 1913 there are no values for n whatever.) The least and greatest integers permissible for forming the squares are represented by another formula of Zerr's, namely,

$$\frac{3828 \pm (n^3 - n)}{2n}.$$

For $n = 3$, the least integer = 634, the greatest integer = 642. For $n = 11$, the least integer = 114, the greatest integer = 234. For $n = 29$, the least integer = $-$ 354, the greatest integer = 486.

Hence the magic squares for 1914 with only positive numbers are as follows:

$n = 3$

641	634	639
636	638	640
637	642	635

* AMERICAN MATHEMATICAL MONTHLY, Vol. XVI, no. 1, page 2.

$n = 11$

181	194	207	220	233	114	127	140	153	166	179
193	206	219	232	124	126	139	152	165	178	180
205	218	231	123	125	138	151	164	177	190	192
217	230	122	135	137	150	163	176	189	191	204
229	121	134	136	149	162	175	188	201	203	216
120	133	146	148	161	174	187	200	202	215	228
132	145	147	160	173	186	199	212	214	227	119
144	157	159	172	185	198	211	213	226	118	131
156	158	171	184	197	210	223	225	117	130	143
168	170	183	196	209	222	224	116	129	142	155
169	182	195	208	221	234	115	128	141	154	167

NOTE ON A MEMORY DEVICE FOR HYPERBOLIC FUNCTIONS.

By F. S. ELDER, Central High School, Kansas City, Mo.

Purely as a memory aid for expressing any hyperbolic function $\phi h(u)$ in terms of any other hyperbolic function $\psi h(u)$, I offer the following method which I never have seen in print.

Draw the customary right triangle ABC right angled at C, as for the circular functions. Write $\psi h(u)$ and unity upon the same sides of the triangle as you would to show the corresponding circular function. Then take the triangle by the point C and deform it until B becomes the right angle. In this deformation, the two sides on which the numbers are written are supposed to retain their original length, while the third side is deformed (lengthened or shortened) and likewise in returning to the original form. Now compute the third side in terms of the other two and let the triangle recover its original shape. All the other hyperbolic functions, $\phi h(u)$, can now be written down directly in terms of $\psi h(u)$ by means of the same ratios of the sides of the original triangle that give the corresponding circular functions.

Examples: (1) If $\psi h(u)$ be $\tan h(u)$, write $AC = 1$, $BC = \tan h(u)$. Deform. Then $AB = \sqrt{1 - \tan^2 h(u)}$; now restore the triangle.

Then

$$\sin h(u) = BC/AB = \frac{\tan h(u)}{\sqrt{1 - \tan^2 h(u)}}, \quad \sec h(u) = AB/AC = \sqrt{1 - \tan^2 h(u)}, \text{ etc.}$$

(2) If $\psi h(u)$ be $\csc h(u)$, write $AB = \csc h(u)$; $BC = 1$. Deform. Then $AC = \sqrt{1 + \csc^2 h(u)}$. Restore.

Then

$$\cos h(u) = AC/AB = \frac{\sqrt{1 + \csc^2 h(u)}}{\csc h(u)}, \quad \tan h(u) = BC/AC = \frac{1}{\sqrt{1 + \csc^2 h(u)}}, \text{ etc.}$$

BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

Mathematik für Biologen und Chemiker. By Professor L. MICHAELIS, Private Docent at the University of Berlin. Springer, Berlin, 1912. vii+253 pages. 7.80 Marks.

This is an interesting example of a type of text similar to the English book by Partington,[1] designed as a minimum course in calculus for students whose engrossing interest is in other than mathematical lines. It is written in clear and "easy" German, and is brought thus to the attention of the readers of the Monthly because its perusal along with other similar texts cannot fail to be of value to those who, either inside or outside the ranks of teachers of mathematics, are concerned in bringing the benefits of the study of calculus to students of general science.

In the first fifty pages the author presents a compendium of the essentials of elementary geometry, arithmetic, algebra and trigonometry; he continues with chapters treating the theory of elementary functions (coördinate geometry), differential calculus (52 pp.), integral calculus (60 pp.), Maclaurin's and Taylor's series (19 pp.), and differential equations (28 pp.). Professor Michaelis frankly adopts the infinitesimal method of exposition *ab initio*, defining dx as an infinitely small increment given to x, adding the statement that dy is similarly the differential of y and that dy/dx is the differential quotient. As a consequence of this style of treatment, the author in his derivation of the differential quotient of the logarithm is spared the logical necessity of proving (or even of mentioning) that "the limit of the logarithm equals the logarithm of the limit."

Numerous applications of calculus to mechanics and chemistry increase the value of the text, although there is little of direct application to biology, as the title would seem to suggest. W. DEW. CAIRNS.

Tables and Formulas for Solving Numerical Problems in Analytic Geometry, Calculus and Applied Mathematics. Arranged by WILLIAM RAYMOND LONGLEY, assistant professor of mathematics in the Sheffield Scientific School, Yale University. Ginn and Company, Boston, 1913. iv+31 pages. $0.50.

The tables in this book comprise the usual four-place trigonometric and logarithmic tables, the equivalents in radians of angles measured in degrees, the natural functions of angles measured in radians, squares, cubes, square roots, cube roots, reciprocals, Naperian logarithms, and exponential and hyperbolic functions.

The formulas from algebra and elementary geometry number twenty, those from trigonometry twenty-two, while there are fifteen from plane and solid analytic geometry and eight from the calculus in addition to twenty-seven formulas for differentiation and one hundred and twenty-seven for integration.

The size of the book ($7\frac{1}{2}$ in. x 5 in.) and its well chosen material should make it popular with those dealing with computational work, especially in technical

[1] *Higher Mathematics for Chemical Students.* By J. R. PARTINGTON. New York, D. Van Nostrand Co., 1912.

schools for which the book was primarily prepared. The typographical clearness is especially to be commended. A. L. UNDERHILL.

Society for the Promotion of Engineering Education, Proceedings of the Twentieth Annual Meeting, June 26–29, 1912, Vol. XX, Part II. Published by the Secretary, Professor H. H. Norris, Ithaca, N. Y. xxii + 508 pages. $1.25 to non-members.

This volume is a splendid example of what may be done to share with all the members of a society and with others outside the society the full benefits of the annual meetings. According to the well-arranged statistics which form the introduction, the membership of the Society numbered 1,166, while only 215 members and guests attended the Boston meeting; yet the report of the treasurer shows that with an annual fee of four dollars the budget makes possible the distribution to members of the full proceedings of the annual meetings as well as of a monthly *Bulletin*. Why may not this be done in more of our American societies than is the case at present, whether these are directly educational or not? It is at times exceedingly difficult to obtain certain papers on a given program, inasmuch as the publishing of these is scattered over a wide range of journals. This *Bulletin* by the way affords an active forum for a live and even spicy discussion of matters bearing particularly on the teaching of engineering subjects, and enables a group of members to carry on during the year an exchange of opinions and a comparison of methods which cannot at all take place with such fullness and deliberation in connection with the all too crowded programs. The practical question fairly forces itself upon our attention as to the possibility of such interchanges on the part of those interested in the teaching of mathematics. A suggestion lies close at hand. Can this be done through the columns of the AMERICAN MATHEMATICAL MONTHLY?

Another valuable feature is the animated discussions which follow the presentation of a set of papers clustering about a common topic, as witness the shorter comments and somewhat more extended criticisms in the *Bulletin* called forth by a paper on "The Teaching of Elementary Physics" by Professors Franklin and MacNutt, or the discussion following a valuable report by Professor W. T. Magruder which collates the results of an investigation of the mechanical engineering laboratories in more than twenty-five institutions in this country with regard to equipment, personnel and methods. The remarks made are always "parliamentary" and in good spirit, but are given now and again "without gloves"; and woe be to the man who has a hobby, or who expresses his opinions either immoderately or vaguely!

Aside from the articles just referred to, there may be singled out for mention papers on "The Faculty Seminar" by Professor H. H. Norris, on "The Hydraulic Equipment at the Ohio State University" by Professor Horace Judd and on "The Engineering Laboratories of the Royal Technical University at Charlottenburg, Germany" by Mr. R. R. Heuter.

The volume may be obtained directly from the Secretary of the Society.

W. DEW. CAIRNS.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

Special Notice.—Please reread the requests as to form of solutions on pp. 258–259 of the October 1913 issue. Unless these directions are observed by contributors, solutions must either be entirely rewritten by the committee or else rejected. Put all drawings on separate sheets.

MANAGING EDITOR.

ALGEBRA.

When this issue was made up no solutions had been received for numbers 396 to 400 inclusive. Please give attention to these.

401. Proposed by R. D. CARMICHAEL, Indiana University.

Prove the validity of Borda's series:

$$\log(x+2) = 2\log(x+1) - 2\log(x-1) + \log(x-2) + 2\left[\frac{2}{x^2-3x} + \frac{1}{3}\left(\frac{2}{x^2-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^2-3x}\right)^5 + \cdots\right].$$

402. Proposed by R. D. CARMICHAEL, Indiana University.

Obtain other series similar to that of Borda, given in the preceding problem.

403. Proposed by C. N. SCHMALL, New York City.

A torpedo-boat 40 miles from shore strikes a rock, making a rent in her hull which admits water at the rate of 15 tons in 48 minutes. The ship's pumps can expel 12 tons in an hour. If 60 tons of water is sufficient to sink the boat, find the average rate of steaming so that it may reach the shore just as it is about to sink.

404. Proposed by V. M. SPUNAR, Chicago, Illinois.

Show that $(a+b)(a+b-1)\cdots(a+b-n+1) = a(a-1)(a-2)\cdots(a-n+1) + \binom{n}{1}a(a-1)(a-2)\cdots(a-n+1)b + \binom{n}{2}a(a-1)(a-2)\cdots(a-n+1)b(b-1) + \cdots + b(b-1)(b-2)\cdots(b-n+1)$.

GEOMETRY.

When this issue was made up no solutions had been received for numbers 417, 421 and 425 to 430 inclusive. Please give attention to these.

The Morley Theorem

431. Proposed by F. M. MORGAN, Dartmouth College.

Trisect the angles of the triangle ABC and let the trisectors nearest each side meet in the respective points M, N, P. Prove by trigonometry that the triangle MNP is equilateral.

432. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Having given the edges of a tetrahedron, a, b, c, d, e, f, find an expression for the radius of the sphere which is tangent to the six edges.

433. Proposed by W. H. BUSSEY, University of Minnesota.

A transformation of the plane keeping the radius of curvature of all curves invariant is either (1) a real or imaginary motion or reflexion, or (2) not a point transformation.

CALCULUS.

When this issue was made up no solutions had been received for numbers 335, 337, 338, 340, 342, 348, 350, and 352.

353. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

The center of a sphere, radius $R = 5$ inches, is $a = 10$ inches above the surface of a sphere, radius $R = 12\frac{1}{2}$ inches. There is a point of light at $b = 1$ inch horizontally from a point $c = 10$ inches vertically above the surface of the first sphere. What is the area of the shadow which the upper sphere casts on the lower one?

354. Proposed by C. N. SCHMALL, New York City.

Prove $\Gamma(1 + x)\Gamma(1 - x) = \dfrac{\pi x}{\sin \pi x}$.

355. Proposed by C. N. SCHMALL, New York City.

Given the curve of the nth degree,

$$y^n - (a + bx)y^{n-1} + (c + dx + ex^2)y^{n-2} + \cdots = 0,$$

show that if each ordinate is divided by the corresponding subtangent, the sum of all the resulting ratios will be a constant.

MECHANICS.

When this issue was made up no solutions had been received for numbers 271, 272, 275, 277, 278, 279, and 282.

286. Proposed by C. N. SCHMALL, New York City.

A slightly elastic string is just long enough to reach between two hooks on the same horizontal line. A ring of weight w is placed at its middle point. Show that the ring will sink through a distance $h = a\sqrt{3ew/2}$, where e is the elasticity of the string and $2a$ the distance between the hooks.

287. Proposed by WALTER H. DRANE, Lebanon, Tenn.

While sitting in an empaled enclosure, I noticed that the spokes of the wheels of passing automobiles, when viewed through the pickets of the fence, appeared to revolve more slowly than they really did, and in some instances even appeared to be revolving in a direction opposite to that in which they were really turning. Explain this optical illusion.

NUMBER THEORY.

When this issue was made up no solutions had been received for numbers 187, 189, 191, 192, 194, 201, and 202 inclusive.

205. Proposed by E. T. BELL, New York City.

Show that in the usual arithmetical sense the form that follows admits of composition; give the requisite transformations, and indicate how several (if not all) solutions may be found. The variables are the x_i.

$$x_0^2 + nrx_1^2 + mrx_2^2 + mnx_3^2 + mnrx_4^2 + mn^2r^2x_5^2 + nr^2m^2x_6^2 + rm^2n^2x_7^2.$$

206. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that the sum of the sixth powers of two integers cannot be the square of an integer.

207. Proposed by A. J. KEMPNER, University of Illinois.

There are 80 positive integers < 100 containing no figure 9 against 19 containing at least one figure 9. (For integers < 1000 the numbers are 728 and 271 respectively.) One might be led to believe that for every positive integer M the number N_1 of positive integers $< M$ containing no figure 9 is always greater than the number N_2 of positive integers $< M$ containing at least one figure 9.

To prove: $\lim_{M=\infty} N_1/N_2 = 0$. See pages 48–50 of January, 1914, issue.

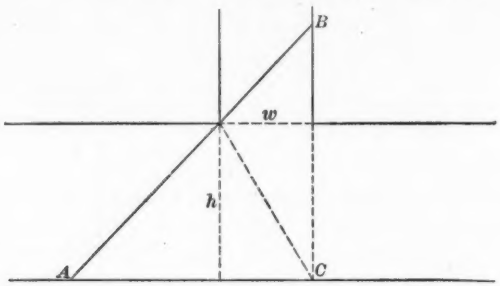
SOLUTIONS OF PROBLEMS.

ALGEBRA.

392. Proposed by H. PRIME, Boston, Mass.

The floor and ceiling of a room h feet high are parallel and horizontal. In the middle of the ceiling is a vertical circular tube w feet in diameter extending upward from the ceiling. From the room the longest possible inflexible rod is put up the tube. When the rod is in contact with the opposite sides of the tube and the floor, what is the expression for the tangent of the acute angle made by the rod and the floor in terms of h and w, considering the rod as an air line? To be solved without the calculus. (From The Maine Farmers' Almanac, 1913.)

I. SOLUTION BY H. E. TREFETHEN, Colby College.



If AB is the rod, $AC = x$ its projection on the floor, and $BC = y$, then

$$x^2 + y^2 = AB^2 = \text{minimum}, \qquad w/x + h/y = 1. \tag{1, 2}$$

Put $w/x = mu$, $x^2 = mu^{-2}$, $h/y = nv$, $y^2 = nv^{-2}$. Then $m = w^{\frac{1}{2}}$, $n = h^{\frac{1}{2}}$. Hence (1, 2) become $w^{\frac{1}{2}}u^{-2} + h^{\frac{1}{2}}v^{-2} = \text{minimum}$, $w^{\frac{1}{2}}u + h^{\frac{1}{2}}v = 1$.

Now

$$(w^{\frac{1}{2}}u^{-2} + h^{\frac{1}{2}}v^{-2})/(w^{\frac{1}{2}} + h^{\frac{1}{2}}) \geqq [(w^{\frac{1}{2}}u + h^{\frac{1}{2}}v)/(w^{\frac{1}{2}} + h^{\frac{1}{2}})]^{-2}. \tag{3}$$

For if

$$w^{\frac{1}{2}}/(w^{\frac{1}{2}} + h^{\frac{1}{2}}) = p, \qquad h^{\frac{1}{2}}/(w^{\frac{1}{2}} + h^{\frac{1}{2}}) = q, \quad \text{then} \quad p + q = 1, \tag{4}$$

and if

$$u/(pu + qv) = r, \qquad v/(pu + qv) = s, \quad \text{then} \quad pr + qs = 1, \tag{5}$$

and if we divide both sides of (3) by the second member, then (3) reduces to $pr^{-2} + qs^{-2} \geqq 1$, which is true since

$$1 - r^2 \geqq 2r^2(1 - r) \quad \text{and} \quad 1 - s^2 \geqq 2s^2(1 - s).$$

And hence

$$r^{-2} - 1 \geqq 2(1 - r) \quad \text{and} \quad pr^{-2} - p \geqq 2p - 2pr, \tag{6}$$

$$s^{-2} - 1 \geqq 2(1 - s) \quad \text{and} \quad qs^{-2} - q \geqq 2q - 2qs. \tag{7}$$

Adding (6, 7) and reducing by (4, 5) we have $pr^{-2} + qs^{-2} \geqq 1$.

Thus (3) is established. Therefore, since the denominators and the second

numerator are constants, $w^{\frac{1}{3}}u^{-2} + h^{\frac{1}{3}}v^{-2}$ = minimum when (3) is an equality, that is, when $u = v$. Hence $w/x = h^{\frac{1}{3}}/y$ and

$$y/x = h^{\frac{1}{3}}/w^{\frac{1}{3}} = \tan BAC \text{ as required.}$$

II. Solution by A. H. Holmes, Brunswick, Me.

A maximum length of pole would move with equal freedom on floor and side of tube, the condition being that the ratio between the part of the pole below the tube to the tangent of the angle it makes with the floor is equal to the ratio between the part of the pole within the tube and the tangent of the angle made by the extremity of the pole with the side of the tube, which is the cotangent of the angle made by the pole with the floor. Let h be the perpendicular height of the ceiling from the floor, w the diameter of the tube, and θ the acute angle made by the pole with the floor. Then we have

$$\frac{h}{\sin\theta} : \tan\theta = \frac{w}{\cos\theta} : \cot\theta.$$

Hence $\tan\theta = \sqrt[3]{h/w}$.

393. Proposed by H. E. Trefethen, Waterville, Maine.

Expand $\log[(\sin x)/x]$ in terms of x.

I. Solution by F. M. Morgan, Dartmouth College.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Hence

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots.$$

Now let

$$\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots = \gamma.$$

Then

$$\frac{\sin x}{x} = 1 - \gamma, \quad \text{and} \quad \log(1-\gamma) = -\gamma - \frac{\gamma^2}{2} - \frac{\gamma^3}{3} - \cdots.$$

Hence

$$\log[(\sin x)/x] = -\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right) - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)^2 - \cdots$$

$$= -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots.$$

II. Solution by A. H. Holmes, Brunswick, Me.

$$\frac{d}{dx}\log[(\sin x)/x] = \cot x - \frac{1}{x}.$$

But

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots.$$

Hence,

$$\frac{d}{dx}\log\left[(\sin x)/x\right] = -\frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots$$

from which by integration, we get

$$\log\left[(\sin x)/x\right] = -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots$$

A solution similar to I was received from LEVY M. COFFIN.

GEOMETRY.

423. Proposed by C. N. SCHMALL, New York City.

The sum of the squares of the distances of a point from n fixed points is constant. Show that the locus of the point is a circle.

SOLUTION BY W. C. EELLS, Tacoma, Wash.

Let (x_1, y_1), (x_2, y_2) $\cdots$ (x_n, y_n) be the coördinates of the n fixed points, and (x, y) of the moving point. Since $d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, the given condition may be expressed,

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + \cdots + (x - x_n)^2 + (y - y_n)^2 = k,$$

which reduces to

$$nx^2 + ny^2 - 2(x_1 + x_2 + \cdots + x_n)x - 2(y_1 + y_2 + \cdots + y_n)y + (x_1^2 + x_2^2 + \cdots + x_n^2) + (y_1^2 + y_2^2 + \cdots + y_n^2) - k = 0.$$

This may be expressed more briefly,

$$x^2 + y^2 - \left(\frac{2}{n}\sum_{i=1}^{n} x_i\right)x - \left(\frac{2}{n}\sum_{i=1}^{n} y_i\right)y + \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k\right) = 0.$$

This is a circle, with center at $\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{i=1}^{n} y_i\right)$, the square of whose radius is

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 + \left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k\right).$$

Similar solutions were received from H. C. FEEMSTER, S. W. REAVES, F. M. MORGAN, ELMER SCHUYLER, and C. N. SCHMALL.

A solution of 418 was received from D. F. KELLEY; one of 420 from C. N. SCHMALL, and one of 422 from H. E. TREFETHEN, after the list of solutions was made up and sent to the printer.

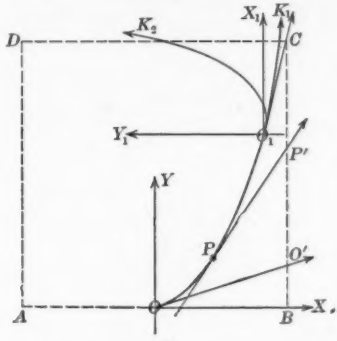
CALCULUS.

327. Proposed by RICHARD P. LOCHNER, Philadelphia, Penn.

A hound is at the middle point of the side of a square field, and a fox is at an adjacent corner. How far will the hound run to catch the fox if the fox runs on the perimeter of the field and the hound runs directly towards the fox at all times, the hound running n times as fast as the fox? Where will the race end?

SOLUTION BY H. EGERER, Wilkinsburg, Pa.

The hound is running on a curve, which has the property that the length of the curve between two points is proportional to the distance cut off by the tangents at these two points on a straight line.



To find this curve $y = f(x)$, which we call K, we take two perpendicular axes through a point O of the curve, so that the y-axis is parallel to the given straight line BC. Furthermore, O', the starting point of the fox, is given by $OB = a$, $O'B = b$. Then we have

$$\text{arc } OP = n \cdot O'P'.$$

Now

$$\text{arc } OP = \int_0^x \sqrt{1 + y'^2}\, dx, \qquad O'P' = y + y'(a - x) - b,$$

so that we have

$$y - b + y'(a - x) = \frac{1}{n}\int_0^x \sqrt{1 + y'^2}\, dx.$$

By differentiation, we get

$$y''(a - x) = \frac{1}{n}\sqrt{1 + y'^2}.$$

If we assume $y' = p$, hence $y'' = \dfrac{dp}{dx}$, we have $\dfrac{dx}{n(a - x)} = \dfrac{dp}{\sqrt{1 + p^2}}$,

and by integration

$$c(a - x)^{1/n} = p + \sqrt{1 + p^2},$$

or

$$p = \frac{dy}{dx} = \frac{c}{2}(a - x)^{1/n} - \frac{1}{2c}(a - x)^{-1/n},$$

and again by integration

$$y = y_a + \frac{n}{2c(n - 1)}(a - x)^{(n-1)/n} - \frac{cn}{2(n + 1)}(a - x)^{(n+1)/n}.$$

Since when $x = y = 0$, $p = \frac{b}{a}$, therefore $c = \frac{b + \sqrt{a^2 + b^2}}{a^{(n+1)/n}}$.

Hence, also

$$y_a = \frac{cn}{2(n+1)} a^{(n+1)/n} - \frac{n}{2c(n-1)} a^{(n-1)/n}.$$

Therefore, finally, we have

$$y = \frac{cn}{2(n+1)} (a^{(n+1)/n} - (a-x)^{(n+1)/n}) - \frac{n}{2c(n-1)} (a^{(n-1)/n} - (a-x)^{(n-1)/n}).$$

We see that the curve K_1 cuts BC at the point (a, y_a) and, since for $n = 1$ $y_a = \infty$, it follows that n must be greater than 1 if the hound can catch the fox.

If we assume the side of the square AB equal to 2, then we have $a = 1$, $b = 0$ and therefore $c = \pm 1$, where we have to take $c = -1$, because y' is always positive and we get

$$y = \frac{n}{2(n-1)} (1 - (1-x)^{(n-1)/n}) - \frac{n}{2(n+1)} (1 - (1-x)^{(n+1)/n}),$$

and

$$y_a = \frac{n}{2(n-1)} - \frac{n}{2(n+1)}.$$

As long as $y_a < 2$, the hound catches the fox on the side BC of the square, and we find the smallest n from $2 = \frac{n}{2(n-1)} - \frac{n}{2(n+1)}$, that is, $n = 1.3680$. If $n < 1.3680$, then we find the point $O_1 \equiv (x_1, y_1)$ whose tangent passes through C and lay a new system of axes X_1Y_1 for the curve K_2. Here is $a = 2 - y_1$, $b = 1 - x_1$, so that we find the equation of the curve K_2 and can find y_a in which alone we are interested. If the hound does not catch the fox on the side CD we have to repeat the process.

We find O_1 by use of the equation $2 = y + y'(a - x)$.

By substituting the above values for y and y', we find

$$\frac{2n}{n^2 - 1} - 4 = \frac{(1-x)^{(n+1)/n}}{n+1} + \frac{(1-x)^{(n-1)/n}}{n-1}.$$

If we assume $(1-x)^{1/n} = U$, we have the form $a = bU^m + cU^{m'}$ from which we find U, and hence x and O_1.

Also solved by BARNEM LIBBY.

334. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve

$$\frac{\partial^2 T}{\partial u \partial v} + \frac{2v}{u^2 + v^2 + 1} \frac{\partial T}{\partial u} + \frac{2u}{u^2 + v^2 + 1} \frac{\partial T}{\partial v} = 0.$$

SOLUTION BY R. D. CARMICHAEL, Indiana University.

The equation may be written in the form

$$\frac{\partial}{\partial v}\left(\frac{\partial T}{\partial u}+\frac{2u}{u^2+v^2+1}T\right)+\frac{2v}{u^2+v^2+1}\left(\frac{\partial T}{\partial v}+\frac{2u}{u^2+v^2+1}T\right)=0.$$

This equation may be viewed as an ordinary differential equation for determining the quantity in parenthesis; the arbitrary "constant" is then a function of u. Solving this equation, we have

$$\frac{\partial T}{\partial u}+\frac{2u}{u^2+v^2+1}T=\frac{\bar{\psi}(u)}{u^2+v^2+1};$$

or

$$(u^2+v^2+1)\frac{\partial T}{\partial u}+2uT=\bar{\psi}(u).$$

Hence

$$(u^2+v^2+1)T=\int\bar{\psi}(u)du+\varphi(v)=\psi(u)+\varphi(v).$$

Therefore, we have

$$T=\frac{\psi(u)+\varphi(v)}{u^2+v^2+1},$$

where $\psi(u)$ and $\varphi(v)$ are arbitrary functions of u and v respectively.

Solved in a different manner by the Proposer.

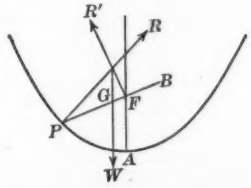
MECHANICS.

267. Proposed by G. H. LIGHT, Purdue University.

A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests; required the position of equilibrium. [From Bowser's Mechanics, Ex. 37, p. 96.]

SOLUTION BY CHRISTIAN HORNUNG, Heidelberg University, Tiffin, O.

Let PB be the beam of length l, of weight W, resting on the peg at the focus, F; let $AF = p$, $AFP = \theta$, and $GP = l/2$. From the polar equation of the parabola we have $FP = 2p/(1+\cos\theta)$.



Since all the surfaces are smooth the resistances, R and R', at P and F respectively, are normal and therefore the angle $BPR = \theta/2$.

Using the equations $\Sigma x = 0$, $\Sigma y = 0$ and Σ(moments about F) $= 0$, which

express the conditions of equilibrium, we have respectively:

$$R\sin\frac{\theta}{2} - R'\cos\theta = 0, \qquad R\cos\frac{\theta}{2} + R'\sin\theta - W = 0,$$

$$R\left(\frac{2p}{1+\cos\theta}\right) - W\left(\frac{2p}{1+\cos\theta} - \frac{l}{2}\right) = 0.$$

From these three equations we get

$$\cos^4\frac{\theta}{2} = \frac{p}{l}, \qquad \cos\frac{\theta}{2} = \left(\frac{p}{l}\right)^{\frac{1}{4}}, \qquad \theta = 2\cos^{-1}\left(\frac{p}{l}\right)^{\frac{1}{4}},$$

which is the result given as the answer to the problem by the author.

Excellent solutions were received from M. E. GRABER, E. B. ESCOTT, J. E. SANDERS, J. SCHEFFER, A. M. HARDING, M. J. MCCUE, MRS. H. E. TREFETHEN, S. G. BARTON and the PROPOSER.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

193. Proposed by W. E. HEAL, Washington, D. C.

Develop a formula for the value of x in the equation $p^x + q^x = s^x$, p, q, and s being integral numbers.

SOLUTION BY B. F. YANNEY, University of Wooster.

By letting $a = p/s$, $b = q/s$, the given equation may be reduced to the form

$$a^x + b^x - 1 = 0.$$

Setting $y = a^x + b^x - 1$, and remembering that

$$a^x = 1 + x\log_e a + \frac{x^2\log_e^2 a}{2!} + \frac{x^3\log_e^3 a}{3!} + \cdots,$$

we have

$$y = 1 + x\log_e(ab) + x^2\frac{\log_e^2 a + \log_e^2 b}{2!} + \cdots.$$

By reversion, we get

$$x = A(y-1) + B(y-1)^2 + C(y-1)^3 + D(y-1)^4 + \cdots,$$

where

$$A = \frac{1}{\log_e(ab)}, \qquad B = -\frac{\log_e^2 a + \log_e^2 b}{2!\log_e^3 a},$$

$$C = \frac{2\cdot 3!\,(\log_e^2 a + \log_e^2 b)^2 - (2!)^2\log_e(ab)\cdot(\log_e^3 a + \log_e^3 b)}{(2!)^2 3!\log_e^5 a},$$

$$D = -\{5\cdot 3!4!\,(\log_e^2 a + \log_e^2 b)^3 - 5\cdot(2!)^2 4!\log_e(ab)\cdot(\log_e^2 a + \log_e^2 b)(\log_e^3 a + \log_e^3 b)$$
$$+ (2!)^3 3!\log_e^2(ab)\cdot(\log_e^4 a + \log_e^4 b)\} \div \{(2!)^3 3!4!\log_e^7 a\},$$

etc.

Now, by hypothesis $y = 0$.

Hence

$$x = -A + B - C + D - \cdots.$$

Also solved by A. H. HOLMES. MR. YANNEY sent a solution of 188 after the list of solutions was sent to the printer.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

QUESTION.

6. In what ways and to what extent will the teaching of mathematics and the content of the curriculum probably be affected by the increasing demand for vocational training?

REPLIES.

2. It is clear, on the most casual observation, that the average college curriculum in mathematics is unbalanced in two respects: (1) algebra predominates over geometry, (2) analytic geometry predominates over our synthetic geometry. This reacts in two undesirable ways: (1) to deprive the college student of a rich and interesting field of study, (2) to give a very one-sided training for prospective teachers of high school mathematics. What can be done to remedy this situation?

IV. Remarks by R. D. Carmichael, Indiana University.

The discussion of this question both by Professor Bussey in the issue of November, 1913, and by several correspondents in the January issue has been stimulating. There can be no doubt that our college curriculum is unbalanced in some important respects. In fact it would probably be hard to give a good reason why certain topics are retained in our courses and still harder to say why others have not been introduced.

It seems clear that the claim of synthetic geometry is a valid one. The beauty of the subject cannot be doubted, and its appeal to the majority of students will probably be granted. It adds to the esthetic delight of one's intellectual life when he is able to picture space as shot through with the beautiful relations which have been created by geometers. It gives him a language of precision and imagery in which to set forth the relations of phenomena in the world of external things through which he must get about in his daily experience.

The purpose of these remarks, however, is not primarily to emphasize these matters, although the present state of our curriculum shows that they are in need of emphasis; it is rather to point out what appears to be a wrong judgment concerning analysis or a wrong emphasis of its difficulties, as expressed in the following words in one of the communications referred to: "There comes a time in the life of every mathematical student when the continual grind of analysis becomes unbearable."

To the present writer analysis is not a "grind" and has never been; on the other hand, it throws over him a peculiar fascination. His deepest intellectual pleasure is the experience of beholding the beautiful threads of connection among these abstract creations of the mind and in witnessing the marvelous and unexpected ways in which they have contributed to our better understanding of the external world and its phenomena.

If this experience were confined to a single individual it should pass unmentioned. But there are not a few who feel the same sense of charm in the

presence of the far-reaching results of analysis, and find in them (directly or indirectly) the chief source of their pleasure in mathematical discipline; and this experience has been theirs through the whole course of their study.

It is doubtless true, however, that many students find analysis anything but pleasant and that this would continue to be their experience although they gave much attention to its cultivation. For these we need synthetic geometry in all its beauty; and not for these only, but for all students who come to us for mathematical training. Let us adjust our courses so that every college student may have the opportunity to go into this rich and interesting field of study.

3. In connection with the theory of the conduction of electricity through gases, one is led to the differential equation

$$y\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 + b\frac{dy}{dx} = cy + d = 0, \tag{1}$$

where a, b, c, d are constants. For unrestricted values of a, b, c, d the solution of this differential equation presents peculiar difficulties, the series solutions obtained by the customary methods having (apparently) too small a range of convergence to be satisfactory from the point of view of electrical theory. The general solution of this equation is wanted in case it can be found. If no general solution is obtained for unrestricted a, b, c, d, it is desirable to know special values of a, b, c, d or special relations among a, b, c, d which make it possible to find the general solution; and this solution is desired in each case.

I. Remarks by X.

A list of some cases in which the differential equation (1) may be solved by elementary means is the following:

1. When $b = c = d = 0$, $a \neq 0, -1$. The general solution in this case is readily found to be

$$y = (\alpha x + \beta)^{\frac{1}{a+1}},$$

where α and β are arbitrary constants.

2. When $b = c = d = 0$, $a = -1$. The general solution is

$$y = e^{\alpha x + \beta},$$

where α and β are arbitrary constants.

3. When $a = c = d = 0$, $b \neq 0$. It is easy to show that the general solution is given by the equation

$$\int_0^{\alpha y} \frac{dt}{\log t} + bx + \beta = 0,$$

where α and β are arbitrary constants.

4. When $a = b = d = 0$. The general solution is

$$y = -\frac{cx^2}{2} + \alpha x + \beta.$$

5. When $a = b = c = 0$, $d \neq 0$. The general solution is given by the equation

$$\int_0^y \frac{dt}{\sqrt{\alpha - 2d \log t}} + x + \beta = 0.$$

6. When $c = d = 0$, $a \neq 0$, $b \neq 0$. The general solution is given by the equation

$$\int_0^y \frac{at^a dt}{bt^a - \alpha} + x + \beta = 0.$$

7. When $a = b = 0$, $c \neq 0$, $d \neq 0$. The general solution is given by the equation

$$\int_0^y \frac{dt}{\sqrt{\alpha - 2ct - 2d \log t}} + x + \beta = 0.$$

Returning now to the general equation (1), put

$$t = \alpha x + \beta,$$

where α and β are any constants except that α is to be different from zero. Then the differential equation becomes

$$y\frac{d^2y}{dt^2} + a\left(\frac{dy}{dt}\right)^2 + \frac{b}{\alpha}\frac{dy}{dt} + \frac{c}{\alpha^2}y + \frac{d}{\alpha^2} = 0. \tag{2}$$

If the original equation (1) has the solution

$$y = f(x, a, b, c, d), \tag{3}$$

then equation (2) has the solution

$$y = f\left(t, a, \frac{b}{\alpha}, \frac{c}{\alpha^2}, \frac{d}{\alpha^2}\right),$$

and hence the original equation (1) has the solution

$$y = f\left(\alpha x + \beta, a, \frac{b}{\alpha}, \frac{c}{\alpha^2}, \frac{d}{\alpha^2}\right). \tag{4}$$

Therefore, *if any solution* (3) *of* (1) *is known, actually involving* x, *a solution exists in the form* (4) *involving two arbitrary constants.* This fact may be of use in obtaining the general solution of (1).

NOTES AND NEWS.

UNDER THE DIRECTION OF FLORIAN CAJORI.

Professor M. Fréchet, of Poitiers, France, is expected to lecture at the University of Illinois during the next academic year.

The Royal Society has awarded the Sylvester medal to J. W. L. Glaisher, F.R.S., for his mathematical researches.

Dr. Karl Boehm, of Heidelberg, succeeds Professor G. Faber as professor of mathematics in the University of Königsberg.

A portrait of Professor Horace Lamb, well known for his researches in mathematical physics, was presented by subscribers to the University of Manchester, where he has filled the chair of mathematics since 1885. The portrait was painted by his son, Henry Lamb, a rising young artist.

The German Mathematical Society recently subscribed 500 francs per year, for five years, in addition to its earlier subscription of 5,000 francs, toward the cost of the publication of Euler's Complete Works. This cost is now estimated at one million francs, which is twice the original estimate.

The Irish Journal of Education for December, 1913, contains the second instalment of the Report of the National Committee of Fifteen on Geometry Syllabus (U. S. A.), which it is reprinting in full. Five hundred copies of this report have recently been purchased by the State Department of New Jersey and placed in the hands of all teachers of geometry in the state.

Dr. Edward L. Dodd, of the University of Texas, contributes to the *Giornale di Matematiche di Battaglini* (Vol. 51, 1913) an interesting article on "A justification of empirical probability based upon an undetermined *a priori* probability." The topic is particularly important in life insurance.

A biography and popular account of the discoveries of T. J. J. See has been written by W. L. Webb. The book includes a statement in non-technical language of See's theories on the creation of the earth and the heavens.

Cosmological speculations receive attention in the January number of *Scientia*, an international review of scientific synthesis, edited by Italian scholars. Professor T. C. Chamberlin, of the University of Chicago, has an article on Planetesimal Hypotheses; the astronomer A. C. D. Crommelin, of Greenwich, writes on the Capture Theory of Satellites; Dr. T. J. J. See discusses the law of nature in celestial evolution.

Sir Robert S. Ball, Lowndean professor of astronomy and geometry in the University of Cambridge, died November 25, 1913, at the age of 73. Along the lines of applied mathematics he contributed a work on *Experimental Mechanics* in 1871, and one on the *Theory of Screws* in 1900, which embodies important researches begun by him in 1870 and published in a series of memoirs. After 1884 he did very little observing, probably on account of trouble with one of his eyes, accidentally injured in his youth and finally removed in 1897.

Dr. Artemas Martin, of Washington, D. C., has recently published in the *Mathematical Magazine* a paper on "The algebraic solution of the famous three point problem." The problem is to determine a point in the plane of a given triangle at which the sides subtend given angles. The two papers read by Dr. Martin at the International Congress of Mathematicians at Cambridge, England,

in 1912 have been published in the proceedings of the congress. The titles of these papers are "On rational right-angled triangles" and "On powers of numbers whose sum is the same power of some number."

Yoshio Mikami read before the Tôkyô Mathematico-Physical Society, and brought out in the *Tôkyô Sûgaku-Buturigakkwai Kizi* (2 S., Vol. VII, No. 9, p. 157), an article on the formula for an arc of a circle, found in the "Kwatsuyō Sampō," a treatise printed in 1712 as a posthumous work of Seki Kōwa and probably composed as early as 1683. Mikami gives a fuller historical and analytical discussion than previously given, of Seki Kōwa's formula,

$$1276900(d-s)^5a^2 = 5107600d^6s - 23835415d^5s^2 + 43470240d^4s^3 - 37997429d^3s^4$$
$$+ 15047062d^2s^5 - 1501025ds^6 - 281290s^7 \cdots,$$

which may be used for the numerical calculation of an arc a of a circle of diameter d, whose altitude is s.

Dr. A. Mitzscherling has produced an interesting book, *Das Problem der Kreisteilung* (Teubner, 1913), which treats the subject historically. It considers the construction of regular polygons, the trisection and multisection of angles, the mechanisms by which these sections can be effected, and also the methods of approximation. Eight different elementary methods of inscribing a regular polygon of 17 sides are given in detail. For multisection 27 different curves are used. Among American writers reviewed in the book are Dexter, J. B. Miller, T. W. Nicholson and E. W. Hyde.

C. F. B. Funk has an article in Schotten's *Zeitschrift* (November, 1913, p. 463), in which he offers a detailed exposition of logarithmic theory along the lines suggested by Felix Klein in his *Elementarmathematik vom höheren Standpunkt*. It is well known that Klein exposes the defects of the current definition of logarithms and suggests that, for school purposes, logarithms be defined geometrically by means of the equilateral hyperbola. He leaves the elaboration of details of his program to the experienced teacher of elementary mathematics. A detailed exposition was given by C. Frenzel in Schotten's *Zeitschrift*, Vol. 44, p. 1, and is now attempted for a second time by Funk.

The question of the adoption of the metric system in the United States and Great Britain will soon become a matter not merely of educational and scientific concern, but also of economic and trade importance. As pointed out in *Nature* (Nov. 27, 1913, p. 384) the adoption of the metric system promises to become, in the near future, a necessity in our trade dealings with China, Japan and Siam. These countries have taken steps to establish that system. In Japan the system is now obligatory for the services of the customs, excepting a few articles, also for the army, for medicine and for electrical work. Russia is taking steps pointing toward the general introduction of the metric system. The rest of Continental Europe has already adopted the metric reform.

The definitive program for the meeting of the International Commission on

the Teaching of Mathematics to be held at Paris in April, 1914, will consist of three sessions daily from April 1 to 4 inclusive:

First day: Session of the Central Committee. Business Session of the Commission. Session of the Mathematical Society of France.

Second day: General opening session. President, L. Poincaré, Director of Secondary Instruction, representing the Minister of Public Instruction. Address of welcome by P. Appell, Dean of the Faculty of Science, member of the Institute. Response by the President of the commission, F. Klein, of Göttingen. Address by the representative of the Minister of Public Instruction. Lecture by E. Borel on "The adaptation of instruction to the progress of science." Lecture by M. D'Ocagne on "The role of mathematics in the engineering sciences." Working session, taking up question A:—"The introduction of the elementary notions of the differential and integral calculus into secondary instruction." General reporter, E. Beke of Budapest.

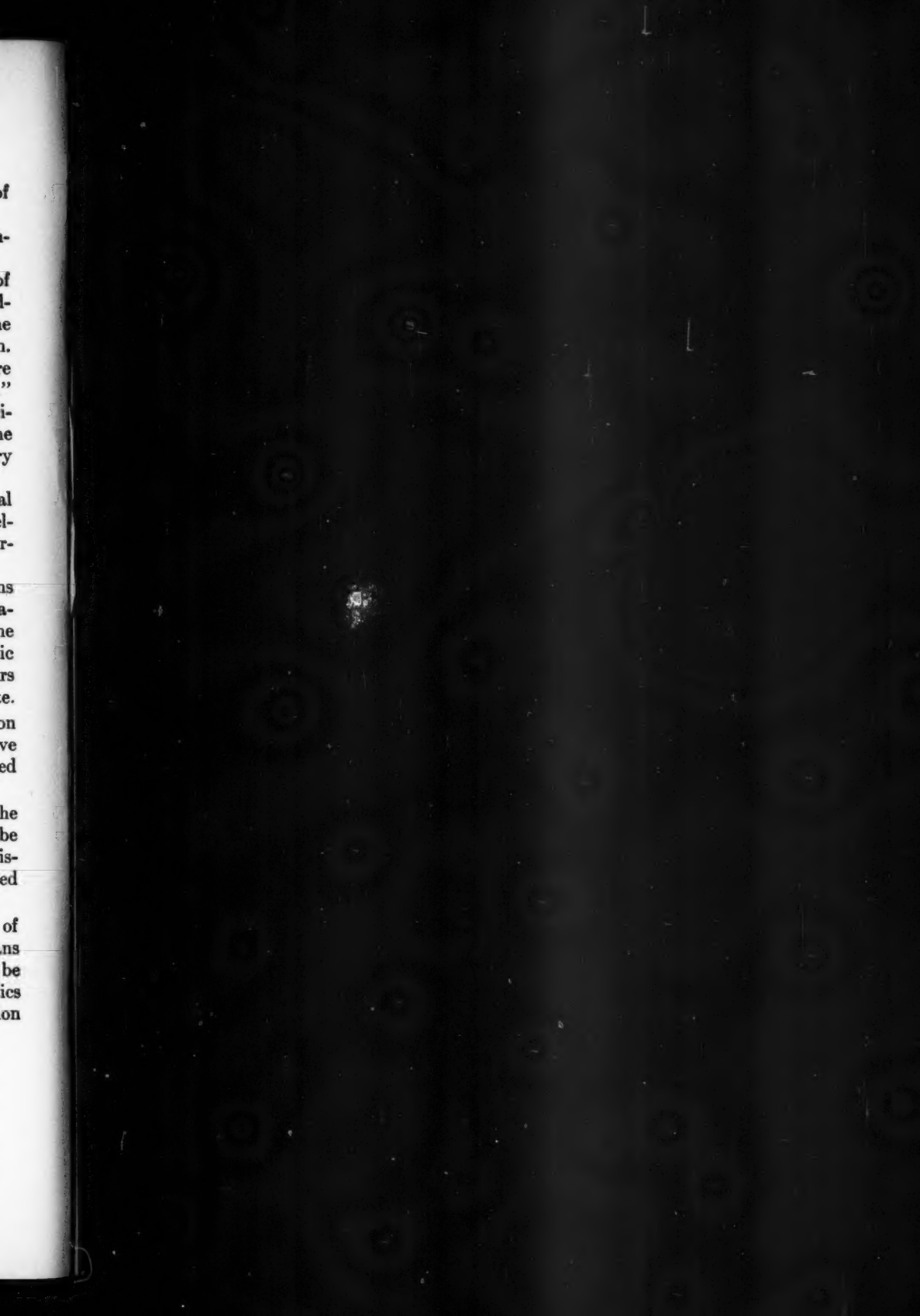
Third day: Working session, taking up question B:—"The mathematical instruction of engineering students." General Reporter, P. Staeckel, of Heidelberg. Working session. Discussion on the teaching of mathematics in engineering schools. Meeting of the Society of Civil Engineers.

Fourth day: Working session. Conclusion of the discussion of questions A and B. Summaries by the general reporters. Business session. Consideration of the future work of the commission, in particular of the program for the meeting of the commission to be held at Munich in 1915, whose principal topic has already been fixed as "The theoretic and practical preparation of instructors of mathematics for the various stages of work." Reception by Prince Bonaparte.

Mr. C. Bourlet, who died last August, has been succeeded in the Commission by C. Bloche, and Messrs. A. de Saint Germain; and C. A. Laisant, who have resigned from the Commission for reasons of health and age, have been replaced by J. Hadamard and M. d'Ocagne.

The sessions will be held at the Sorbonne except as otherwise specified. The general opening session will be public. Admission to the working sessions will be limited to members of the Commission and of the various National Subcommissions and Committees, and to such other persons as shall have been furnished with tickets of admission by the General Secretary.

The Philosophical Society of France, in conjunction with the publishers of the Encyclopedia of the Mathematical Sciences, invites the mathematicians present in Paris on the occasion of this Congress to a series of sessions to be held April 6–8, at which various questions on the philosophy of mathematics will be considered. The Physical Society of France will hold its annual session and exposition of recent apparatus at Paris, April 15–17.



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OPTICAL INTERPRETATIONS IN HIGHER GEODESY.[1]

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It is the object of this paper to show that certain curves which are defined in higher geodesy may be given simple optical interpretations. In order to be able to state what these interpretations are it will be necessary to define certain terms in both higher geodesy and optics.

§ 1. **Terms in Higher Geodesy.** Let us assume a set of rectangular axes O–x, y, z which are at rest with respect to the solid part of the rotating earth and such that the axis Oz is coincident with the earth's axis of rotation, the positive direction of Oz being from the celestial south pole to the celestial north pole, and the positive directions of Ox and Oy being such that $+Oy$ may be obtained from $+Ox$ by rotating $+Ox$ through 90° in the direction in which the earth rotates. Let us denote by P_0, (x_0, y_0, z_0) a point (on or near the earth's surface) which is at rest with respect to these axes. The field of force which determines weight, that is, the field in which a plumb-line is in equilibrium, is at rest with respect to these axes. The straight line which passes through P_0 and gives the direction of the force of this field at P_0, is defined as the *vertical at* P_0. The vertical at P_0 coincides with the string of a plumb-line,[2] the plumb-bob of which is situated at P_0. The *astronomical latitude at* P_0 is the complement ϕ of the angle which the vertical at P_0 (in the direction of the zenith) makes with the earth's axis (in the direction of the celestial north pole). The vertical at a general point P_0 does not intersect the earth's axis of rotation.

The *astronomical meridian plane at* P_0 is the plane which passes through the vertical at P_0 and is parallel to the earth's axis of rotation. The *astronomical longitude at* P_0 is the angle λ which the meridian plane at P_0 makes with a fixed plane through the axis of rotation (we here assume that the fixed plane is the plane zOx) measured from 0° to 360° in the direction in which the earth rotates.

[1] Presented to the American Mathematical Society (Chicago Section), April 5, 1912.

[2] It is assumed that the string of the plumb-line is weightless and perfectly flexible, and that the plumb-bob is a heavy particle.